

Definable and invariant types in enrichments of NIP theories

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Let T be an NIP \mathcal{L} -theory and \tilde{T} be an enrichment. We give a sufficient condition on \tilde{T} for the underlying \mathcal{L} -type of any definable (respectively invariant) type over a model of \tilde{T} to be definable (respectively invariant). These results are then applied to Scanlon's model completion of valued differential fields.

Let T be a theory in a language \mathcal{L} and consider an expansion $T \subseteq \tilde{T}$ in a language $\tilde{\mathcal{L}}$. In this paper, we wish to study how invariance and definability of types in T relate to invariance and definability of types in \tilde{T} . More precisely, let $\mathfrak{U} \models \tilde{T}$ be a monster model and consider some type $\tilde{p} \in \mathcal{S}(\mathfrak{U})$ which is invariant over some small $M \models \tilde{T}$. Then the reduct p of \tilde{p} to \mathcal{L} is of course invariant under the action of the $\tilde{\mathcal{L}}$ -automorphisms of \mathfrak{U} that fix M (which we will denote as $\tilde{\mathcal{L}}(M)$ -invariant), but there is, in general, no reason for it to be $\mathcal{L}(M)$ -invariant. Similarly, if \tilde{p} is $\tilde{\mathcal{L}}(M)$ -definable, p might not be $\mathcal{L}(M)$ -definable.

When T is stable, and $\varphi(x; y)$ is an \mathcal{L} -formula, φ -types are definable by Boolean combinations of instances of φ . It follows that if \tilde{p} is $\tilde{\mathcal{L}}(M)$ -invariant then p is both $\mathcal{L}(M)$ -invariant and $\mathcal{L}(M)$ -definable. Nevertheless, when T is only assumed to be NIP, then this is not always the case. For example one can take T to be the theory of dense linear orders and $\tilde{\mathcal{L}} = \{\leq, P(x)\}$ where $P(x)$ is a new unary predicate naming a convex non-definable subset of the universe. Then there is a definable type in \tilde{T} lying at some extremity of this convex set whose reduct to $\mathcal{L} = \{\leq\}$ is not definable without the predicate.

In the first section of this paper, we give a sufficient condition (in the case where T is NIP) to ensure that any $\tilde{\mathcal{L}}(M)$ -invariant (resp. definable) \mathcal{L} -type p is also $\mathcal{L}(M)$ -invariant (resp. definable). The condition is that there exists a model M of \tilde{T} whose reduct to \mathcal{L} is uniformly stably embedded in every elementary extension of itself. In the case where T is o-minimal for example, this happens whenever the ordering on M is complete.

The main technical tool developed in this first section is the notion of external separability (Definition (1.2)). Two sets X and Y are said to be externally separable if there exists

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1 External separability

an externally definable set Z such that $X \subseteq Z$ and $Y \cap Z = \emptyset$. In Proposition (1.3), we show that in NIP theory, external separability is essentially a first order property. The results about definable and invariant sets then follow by standard methods along with a “local representation” of φ -types from [Sim15b].

The motivation for these results comes from the study of expansions of ACVF and in particular the model completion $\text{VDF}_{\mathcal{EC}}$ defined by Scanlon [Sca00] of valued differential fields with a contractive derivation, i.e. a derivation ∂ such that for all x , $\text{val}(\partial(x)) \geq \text{val}(x)$. In the third section, we deduce, from the previous abstract results, a characterisation of definable (resp. invariant) types in models of $\text{VDF}_{\mathcal{EC}}$ in terms of the definability (resp. invariance) of the underlying ACVF-type. This characterisation also allows us to control the canonical basis of definable types in $\text{VDF}_{\mathcal{EC}}$, an essential step in proving elimination of imaginaries for that theory in [Rid].

0.1 Notation

Let us now define some notation that will be used throughout the paper. When $\varphi(x; y)$ is a formula, we implicitly consider that y is a parameter of the formula and we define $\varphi^{\text{opp}}(y; x)$ to be equal to $\varphi(x; y)$.

We write $N <^+ M$ to denote that M is a $|N|^+$ -saturated and (strongly) $|N|^+$ -homogeneous elementary extension of N .

Let X be an $\mathcal{L}(M)$ -definable set (or a union of definable sets) and $A \subseteq M$. We denote by $X(A)$ the set of realisations of X in A , i.e. the set $\{a \in A : M \models a \in X\}$. If \mathcal{R} is a set of definable sets (in particular a set of sorts), we define $\mathcal{R}(A) := \bigcup_{R \in \mathcal{R}} R(A)$.

1 External separability

Definition 1.1 (Externally φ -definable):

Let M be an \mathcal{L} -structure, $\varphi(x; t)$ be an \mathcal{L} -formula and X a subset of some Cartesian power of M . We say that X is externally φ -definable if there exist $N \geq M$ and a tuple $a \in N$ such that $X = \varphi(M; a)$.

Definition 1.2 (Externally φ -separable):

Let M be an \mathcal{L} -structure, $\varphi(x; t)$ be an \mathcal{L} -formula and X, Y be subsets of some Cartesian power of M . We say that X and Y are externally φ -separable if there exist $N \geq M$ and a tuple $a \in N$ such that $X \subseteq \varphi(M; a)$ and $Y \cap \varphi(M; a) = \emptyset$.

We will say that X and Y are φ -separable if a can be chosen in M . Note that a set X is externally φ -definable if X and its complement are externally φ -separable.

Proposition 1.3:

Let T be an \mathcal{L} -theory and $\varphi(x; t)$ an NIP \mathcal{L} -formula. Let $U(x)$ and $V(x)$ be new predicate symbols and let $\mathcal{L}_{U,V} := \mathcal{L} \cup \{U, V\}$. Then, there is an $\mathcal{L}_{U,V}$ -sentence $\theta_{U,V}$ and an \mathcal{L} -formula $\psi(x; s)$ such that for all $M \models T$ and any enrichment $M_{U,V}$ of M to $\mathcal{L}_{U,V}$, we

1 External separability

have:

if U and V are externally φ -separable, then $M_{U,V} \models \theta_{U,V}$

and

if $M_{U,V} \models \theta_{U,V}$, then U and V are externally ψ -separable.

Proof. Let k_1 be the VC-dimension of $\varphi(x; t)$. By the dual version of the (p, q) -theorem (see [Mat04] and [Sim15a, Corollary 6.13]) there exists q_1 and n_1 such that for any set X , any finite $A \subseteq X$ and any $\mathcal{S} \subseteq \mathcal{P}(X)$ of VC-dimension at most k_1 , if for all $A_0 \subseteq A$ of size at most q_1 there exist $S \in \mathcal{S}$ containing A_0 , then there exists $S_1 \dots S_{n_1} \in \mathcal{S}$ such that $A \subseteq \bigcup_{i \leq n_1} S_i$. Let k_2 be the VC-dimension of $\bigcup_{i=1}^{n_1} \varphi(x; t_i)$ and q_2 and n_2 the bounds obtained by the dual (p, q) -theorem for families of VC-dimension at most k_2 . Let

$$\theta_{U,V} := \forall x_1 \dots x_{q_1}, y_1 \dots y_{q_2} \bigwedge_{i \leq q_1} U(x_i) \wedge \bigwedge_{j \leq q_2} V(y_j) \Rightarrow \exists t \bigwedge_{i \leq q_1} \varphi(x_i; t) \wedge \bigwedge_{j \leq q_2} \neg \varphi(y_j; t).$$

Now, let $M \prec^+ N \models T$, U and V be subsets of $M^{|x|}$ and $d \in N$ be a tuple. If $U \subseteq \varphi(M; d)$ and $V \subseteq \neg \varphi(M; d)$ then for any $A \subseteq U$ and $B \subseteq V$ finite there exists $d_0 \in M$ such that $A \subseteq \varphi(M; d_0)$ and $B \subseteq \neg \varphi(M; d_0)$. In particular, $M_{U,V} \models \theta_{U,V}$.

Suppose now that $M_{U,V} \models \theta_{U,V}$. Let $B_0 \subseteq V$ have cardinality at most q_2 . The family $\{\varphi(M; d) : d \in M \text{ a tuple and } B_0 \subseteq \neg \varphi(M; d)\}$ has VC-dimension at most k_1 (a subfamily always has lower VC-dimension). Because $M_{U,V} \models \theta_{U,V}$, for any $A_0 \subseteq U$ of size at most q_1 , there exists $d \in M$ such that $A_0 \subseteq \varphi(M; d)$ and $B_0 \subseteq \neg \varphi(M; d)$. It follows that for any finite $A \subseteq U$ there are tuples $d_1 \dots d_{n_1} \in M$ such that $A \subseteq \bigvee_{i \leq n_1} \varphi(M; d_i)$ and for all $i \leq n_1$, $B_0 \subseteq \neg \varphi(M; d_i)$, in particular, $B_0 \subseteq \neg (\bigvee_{i \leq n_1} \varphi(M; d_i))$. By compactness, there exists tuple $d_1 \dots d_{n_1} \in N$ such that $U \subseteq \bigvee_{i \leq n_1} \varphi(M; d_i)$ and $B_0 \subseteq \neg (\bigvee_{i \leq n_1} \varphi(M; d_i))$.

The family $\{\neg (\bigvee_{i \leq n_1} \varphi(M; d_i)) : d_i \in N \text{ tuples and } U \subseteq \bigvee_{i \leq n_1} \varphi(M; d_i)\}$ has VC-dimension at most k_2 . We have just shown that for any B_0 of size at most q_2 , there is an element of that family containing B_0 . It follows by the (p, q) -property and compactness that there exists tuples $d_{i,j} \in N \succ M$ such that $V \subseteq \bigvee_{j \leq n_2} \neg (\bigvee_{i \leq n_1} \varphi(M; d_{i,j})) = \neg (\bigwedge_{j \leq n_2} \bigvee_{i \leq n_1} \varphi(M; d_{i,j}))$ and $U \subseteq \bigwedge_{j \leq n_2} \bigvee_{i \leq n_1} \varphi(M; d_{i,j})$. Hence U and V are externally $\bigwedge_{j \leq n_2} \bigvee_{i \leq n_1} \varphi(x; t_{i,j})$ -separable. \blacksquare

We would now like to characterise enrichments \tilde{T} of NIP theories that do not add new externally separable definable sets, i.e. $\tilde{\mathcal{L}}$ -definable sets that are externally \mathcal{L} -separable but not internally \mathcal{L} -separable. We show that if there is one model of \tilde{T} where this property holds uniformly, then it holds in all models of T .

Proposition 1.4:

Let T be an NIP \mathcal{L} -theory (with at least two constants), $\tilde{\mathcal{L}} \supseteq \mathcal{L}$ be some language, $\tilde{T} \supseteq T$ be a complete $\tilde{\mathcal{L}}$ -theory and $\chi_1(x; s)$ and $\chi_2(x; s)$ be $\tilde{\mathcal{L}}$ -formulas. The following are equivalent:

- (i) For all \mathcal{L} -formulas $\varphi(x; t)$, all $M \models \tilde{T}$ and all $a \in M$ there exists an \mathcal{L} -formula $\xi(x; z)$ such that if $\chi_1(M; a)$ and $\chi_2(M; a)$ are externally φ -separated then they are ξ -separated;

1 External separability

- (ii) For all \mathcal{L} -formulas $\varphi(x; t)$, there exists an \mathcal{L} -formula $\xi(x; z)$ such that for all $M \models \tilde{T}$ and all $a \in M$, if $\chi_1(M; a)$ and $\chi_2(M; a)$ are externally φ -separated then they are ξ -separated;
- (iii) For all \mathcal{L} -formulas $\varphi(x; t)$, there exists an \mathcal{L} -formula $\xi(x; z)$ and $M \models \tilde{T}$ such that for all $a \in M$, if $\chi_1(M; a)$ and $\chi_2(M; a)$ are externally φ -separated then they are ξ -separated.

Proof. The implications (ii) \Rightarrow (i) and (ii) \Rightarrow (iii) are trivial.

Let us now show that (iii) implies (ii). By Proposition (1.3), there exists an $\tilde{\mathcal{L}}$ -formula $\theta(s)$ and an \mathcal{L} -formula $\psi(x; u)$ such that for all $N \models \tilde{T}$ and $a \in N$:

$$\chi_1(N; a) \text{ and } \chi_2(N; a) \text{ externally } \varphi\text{-separated implies } N \models \theta(a)$$

and

$$N \models \theta(a) \text{ implies } \chi_1(N; a) \text{ and } \chi_2(N; a) \text{ externally } \psi\text{-separated.}$$

Let M and ξ be as in condition (iii) with respect to ψ . We have:

$$M \models \forall s \theta(s) \Rightarrow \exists u (\forall x (\chi_1(x; s) \Rightarrow \xi(x; u)) \wedge (\chi_2(x; s) \Rightarrow \neg \xi(x; u))).$$

As \tilde{T} is complete, this must hold in any $N \models \tilde{T}$. Thus, if $\chi_1(N; a)$ and $\chi_2(N; a)$ are externally φ -separated, we have $N \models \theta(a)$ and hence $\chi_1(N; a)$ and $\chi_2(N; a)$ are ξ -separated. There remains to prove that (i) \Rightarrow (iii). Pick any $M <^+ \mathfrak{U} \models \tilde{T}$. By (i), it is impossible to find, in any elementary extension (\mathfrak{U}^*, M^*) of the pair (\mathfrak{U}, M) , a tuple $a \in M^*$ and $b \in \mathfrak{U}^*$ such that $\chi_1(M^*; a)$ and $\chi_2(M^*; a)$ are separated by $\varphi(M^*; b)$, but they are not separated by any set of the form $\xi(M^*; c)$ where ξ is an \mathcal{L} -formula and $c \in M^*$. By compactness, there exists $\xi_i(x; u_i)$ for $i \leq n$ such that for all $a \in M$ if $\chi_1(M; a)$ and $\chi_2(M; a)$ are externally φ -separated, there exists an i such that they are ξ_i -separated. By classic coding tricks, we can ensure that $i = 1$. \blacksquare

Definition 1.5 (Uniform stable embeddedness):

Let M be an \mathcal{L} -structure and $A \subseteq M$. We say that A is uniformly stably embedded in M if for all formulas $\varphi(x; t)$ there exists a formula $\chi(x; s)$ such that for all tuples $b \in M$ there exists a tuple $a \in A$ such that $\varphi(A, b) = \chi(A, a)$.

Remark 1.6:

If there exists $M \models \tilde{T}$ such that $M|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension, then such an M witnesses Condition 1.4.(iii) for every choice of formulas χ_1 and χ_2 .

Corollary 1.7:

Let T be an NIP \mathcal{L} -theory that eliminates imaginaries, $\tilde{\mathcal{L}} \supseteq \mathcal{L}$ be some language and $\tilde{T} \supseteq T$ be a complete $\tilde{\mathcal{L}}$ -theory. Suppose that there exists $M \models \tilde{T}$ such that $M|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension. Let $\varphi(x; t)$ be an \mathcal{L} -formula, $N \models \tilde{T}$, $A = \text{dcl}_{\tilde{\mathcal{L}}}^{\text{eq}}(A) \subseteq N^{\text{eq}}$ and $p \in \mathcal{S}_x^{\varphi}(N)$. If p is $\tilde{\mathcal{L}}^{\text{eq}}(A)$ -definable, then it is in fact $\mathcal{L}(\mathcal{R}(A))$ -definable where \mathcal{R} denotes the set of all \mathcal{L} -sorts.

1 External separability

Proof. Let $a \models p$. Then $X := \{m \in N : \varphi(x; m) \in p\} = \{m \in N : \models \varphi(a; m)\}$ is \mathcal{L} -externally definable and $\tilde{\mathcal{L}}^{\text{eq}}(A)$ -definable (by some $\tilde{\mathcal{L}}$ -formula χ). It follows from Remark (1.6) that Condition 1.4.(iii) holds and hence, by Condition 1.4.(i), taking $\chi_1 = \chi$ and $\chi_2 = \neg\chi$, it follows that X is \mathcal{L} -definable.

Because T eliminates imaginaries, we have just shown that we can find $\ulcorner X \urcorner^{\mathcal{L}} \in \mathcal{R}$. But X is also $\tilde{\mathcal{L}}^{\text{eq}}(A)$ -definable, hence any $\tilde{\mathcal{L}}^{\text{eq}}(A)$ -automorphism of N^{eq} stabilises $X(N)$ globally and therefore fixes $\ulcorner X \urcorner^{\mathcal{L}}$. If we assume that N is strongly $|A|^+$ -homogeneous (and we can), it follows that $\ulcorner X \urcorner^{\mathcal{L}} \in \text{dcl}_{\tilde{\mathcal{L}}}^{\text{eq}}(A) = A$. Thus $\ulcorner X \urcorner^{\mathcal{L}} \in A \cap \mathcal{R} = \mathcal{R}(A)$ and X is $\mathcal{L}(\mathcal{R}(A))$ -definable. \blacksquare

We will need the following result, which is [Sim15b, Proposition 2.11].

Proposition 1.8:

Let T be any theory, $\varphi(x; y)$ an NIP formula, $M <^+ N \models T$ and $p(x)$ a global M -invariant φ -type. Let $b, b' \in \mathfrak{U} \succ N$ such that both $\text{tp}(b/N)$ and $\text{tp}(b'/N)$ are finitely satisfiable in M and $\text{tp}_{\varphi^{\text{opp}}}(b/N) = \text{tp}_{\varphi^{\text{opp}}}(b'/N)$. Then we have $p|_{\mathfrak{U}} \vdash \varphi(x; b) \iff \varphi(x; b')$.

Proposition 1.9:

Let T be an NIP \mathcal{L} -theory, $\tilde{\mathcal{L}} \supseteq \mathcal{L}$ be some language and $\tilde{T} \supseteq T$ be a complete $\tilde{\mathcal{L}}$ -theory. Let \mathcal{R} denote the set of \mathcal{L} -sorts. Suppose that there exists $M \models \tilde{T}$ such that $M|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension. Let $\varphi(x; t)$ be an \mathcal{L} -formula, $N \models \tilde{T}$ be sufficiently saturated, $A \subseteq N$ and $p \in \mathcal{S}_x^{\varphi}(N)$ be $\tilde{\mathcal{L}}(A)$ -invariant. Assume that every $\tilde{\mathcal{L}}(A)$ -definable set (in some Cartesian power of \mathcal{R}) is consistent with some global $\mathcal{L}(\mathcal{R}(A))$ -invariant type. Then p is $\mathcal{L}(\mathcal{R}(A))$ -invariant.

Proof. Let us first assume that $A \models \tilde{T}$. Let b_1 and b_2 be such $p(x) \vdash \varphi(x, b_1) \wedge \neg\varphi(x, b_2)$. We have to show that $\text{tp}_{\mathcal{L}}(b_1/A) \neq \text{tp}_{\mathcal{L}}(b_2/A)$. Let $p_i = \text{tp}_{\tilde{\mathcal{L}}}(b_i/A)$, $\Sigma(t)$ be the set of $\tilde{\mathcal{L}}(N)$ -formulas $\theta(t)$ such that $\neg\theta(A) = \emptyset$ and $\Delta(t_1, t_2)$ be the set:

$$p_1(t_1) \cup p_2(t_2) \cup \Sigma(t_1) \cup \Sigma(t_2) \cup \{\varphi(n, t_1) \iff \varphi(n, t_2) : n \in N\}.$$

If Δ were consistent, there would exist b_1^* and b_2^* such that $b_i \equiv_{\tilde{\mathcal{L}}(A)} b_i^*$, $\text{tp}_{\tilde{\mathcal{L}}}(b_i^*/N)$ is finitely satisfiable in A and $\text{tp}_{\varphi^{\text{opp}}}(b_1^*/N) = \text{tp}_{\varphi^{\text{opp}}}(b_2^*/N)$. Applying Proposition (1.8) it would follow that $p(x) \vdash \varphi(x; b_1^*) \iff \varphi(x; b_2^*)$. But, because p is $\tilde{\mathcal{L}}(A)$ -invariant and $p(x) \vdash \varphi(x, b_1) \wedge \neg\varphi(x, b_2)$, we also have that $p(x) \vdash \varphi(x; b_1^*) \wedge \neg\varphi(x; b_2^*)$, a contradiction. By compactness, there exists $\psi_i \in p_i$, $\theta_i \in \Sigma$, $n \in \omega$ and $(c_i)_{i \in n} \in N$ such that

$$\forall t_1, t_2 \theta_1(t_1) \wedge \theta_2(t_2) \wedge (\bigwedge_i \varphi(c_i, t_1) \iff \varphi(c_i, t_2)) \wedge \psi_1(t_1) \Rightarrow \neg\psi_2(t_2).$$

In particular, because $\neg\theta_i(A) = \emptyset$, for all m_1 and $m_2 \in A$, $(\bigwedge_i \varphi(c_i, m_1) \iff \varphi(c_i, m_2)) \wedge \psi_1(m_1) \Rightarrow \neg\psi_2(m_2)$. For all $\varepsilon : n \rightarrow 2$, let $\varphi_{\varepsilon}(t, c) := \bigwedge_i \varphi(c_i, t)^{\varepsilon(i)}$ where $\varphi^1 = \varphi$ and $\varphi^0 = \neg\varphi$. It follows that if $\varphi_{\varepsilon}(A, c) \cap \psi_1(A) \neq \emptyset$, then $\varphi_{\varepsilon}(A, c) \cap \psi_2(A) = \emptyset$. Let

$$\theta(t, c) := \bigvee_{\varphi_{\varepsilon}(A, c) \cap \psi_1(A) \neq \emptyset} \varphi_{\varepsilon}(c, t).$$

2 Valued differential fields

We have $\psi_1(A) \subseteq \theta(A, c)$ and $\psi_2(A) \cap \theta(A, c) = \emptyset$, i.e. $\psi_1(A)$ and $\psi_2(A)$ are externally θ -separable. By Proposition (1.4) and Remark (1.6), $\psi_1(A)$ and $\psi_2(A)$ are in fact ξ -separable for some $\mathcal{L}(\mathcal{R}(A))$ -formula ξ . It follows that $N \models \forall t_1, t_2 (\psi_1(t_1) \Rightarrow \xi(t_1)) \wedge (\psi_2(t_2) \Rightarrow \neg \xi(t_2))$ and, in particular $N \models \xi(b_1) \wedge \neg \xi(b_2)$. So $\text{tp}_{\mathcal{L}}(b_1/A) \neq \text{tp}_{\mathcal{L}}(b_2/A)$.

Let us now conclude the proof when A is not a model. Let $M \models \tilde{T}$ contain A and pick any a and $b \in N$ such that $a \equiv_{\mathcal{L}(\mathcal{R}(A))} b$.

Claim 1.10: *There exists $M^* \equiv_{\tilde{\mathcal{L}}(A)} M$ (in particular it is a model of \tilde{T} containing A) such that $a \equiv_{\mathcal{L}(\mathcal{R}(M^*))} b$.*

Proof. By compactness, it suffices, given $\chi(y, z) \in \text{tp}_{\tilde{\mathcal{L}}}(M/A)$, where y is a tuple of \mathcal{R} -variables, and $\psi_i(t; y)$ a finite number of \mathcal{L} -formulas, to find tuples m, n such that $\models \chi(m, n) \wedge \bigwedge_i \psi(a; m) \iff \psi(b; m)$. By hypothesis on A , there exists $q \in \mathcal{S}_y(N|_{\mathcal{L}})$ which is $\mathcal{L}(\mathcal{R}(A))$ -invariant and consistent with $\exists z \chi(y, z)$. Let $m \models q|_{\mathcal{R}(A)_{ab} \cup \{\chi(y)\}}$. Then $\text{tp}_{\mathcal{L}}(a/m) = \text{tp}_{\mathcal{L}}(b/m)$ and $\models \exists z \chi(m, z)$. In particular, we can also find n . \blacklozenge

As p is $\tilde{\mathcal{L}}(A)$ -invariant it is in particular $\tilde{\mathcal{L}}(M^*)$ -invariant. But, as shown above, p is then $\mathcal{L}(\mathcal{R}(M^*))$ -invariant. It follows that $p \vdash \varphi(x; a) \iff \varphi(x; b)$. \blacksquare

The assumption that all $\tilde{\mathcal{L}}(A)$ -definable sets are consistent with some global $\mathcal{L}(A)$ -invariant type may seem like a surprising assumption. Nevertheless, considering a coheir (in the sense of \tilde{T} , whose restriction to \mathcal{L} is also a coheir in the sense of T), this assumption always holds when A is a model of \tilde{T} .

2 Valued differential fields

The main motivation for the results in the previous sections was to understand definable and invariant types in valued differential fields and more specifically those with a contractive derivation, i.e. for all x , $\text{val}(\partial(x)) \geq \text{val}(x)$. In [Sca00], Scanlon showed that the theory of valued fields with a valuation preserving derivation has a model completion named $\text{VDF}_{\mathcal{EC}}$. It is the theory of ∂ -Henselian fields whose residue field is a model of DCF_0 , whose value group is divisible and such that for all x there exists a y with $\partial(y) = 0$ and $\text{val}(y) = \text{val}(x)$.

The main result that we will be needing here is that the theory $\text{VDF}_{\mathcal{EC}}$ eliminates quantifiers in the one sorted language $\mathcal{L}_{\partial, \text{div}}$ consisting of the language of rings enriched with a symbol ∂ for the derivation and a symbol $x|y$ interpreted as $\text{val}(x) \leq \text{val}(y)$. This result implies that for all substructures $A \leq M \models \text{VDF}_{\mathcal{EC}}$ the map sending $p = \text{tp}_{\mathcal{L}_{\partial, \text{div}}}(c/A)$ to $\nabla_{\omega} p := \text{tp}_{\mathcal{L}_{\text{div}}}((\partial^i(c))_{i \in \omega}/A)$ is injective, where $\mathcal{L}_{\text{div}} := \mathcal{L}_{\partial, \text{div}} \setminus \{\partial\}$ denotes the one sorted language of valued fields.

Lemma 2.1:

Let $k \models \text{DCF}_0$. The Hahn field $k((t^{\mathbb{R}}))$, with derivation $\partial(\sum_i a_i t^i) = \sum_i \partial(a_i) t^i$ and its natural valuation, is a models of $\text{VDF}_{\mathcal{EC}}$ and its reduct to \mathcal{L}_{div} is uniformly stably embedded in every elementary extension.

2 Valued differential fields

Proof. The fact that $k((t^{\mathbb{R}})) \models \text{VDF}_{\mathcal{EC}}$ follows from the fact that its residue field k is a model of DCF_0 , its value group \mathbb{R} is a divisible ordered Abelian group and that Hahn fields are spherically complete, cf. [Sca00, Proposition 6.1].

The fact that $k((t^{\mathbb{R}}))$ is uniformly stably embedded in every elementary extension is shown in [Rid, Corollary A.7]. \blacksquare

Recall that Haskell, Hrushovski and Macpherson [HHM06] showed that algebraically closed valued fields eliminate imaginaries provided the geometric sorts are added. We will be denoting by \mathcal{G} the set of all geometric sorts.

Proposition 2.2:

Let $A = \text{acl}_{\mathcal{L}_{\partial, \text{div}}}^{\text{eq}}(A) \subseteq M \models \text{VDF}_{\mathcal{EC}}$. A type $p \in \mathcal{S}^{\mathcal{L}_{\text{div}}}(M)$ is $\mathcal{L}_{\partial, \text{div}}^{\text{eq}}(A)$ -definable if and only if it is $\mathcal{L}_{\text{div}}^{\text{eq}}(\mathcal{G}(A))$ -definable.

Proof. If p is $\mathcal{L}_{\text{div}}^{\text{eq}}(\mathcal{G}(A))$ -definable then it is in particular $\mathcal{L}_{\partial, \text{div}}^{\text{eq}}(A)$ -definable. The reciprocal implication follows immediately from Corollary (1.7) and Lemma (2.1). \blacksquare

An immediate corollary of this proposition is an elimination of imaginaries result for canonical bases of definable types in $\text{VDF}_{\mathcal{EC}}$:

Corollary 2.3:

Let $A = \text{acl}_{\mathcal{L}_{\partial, \text{div}}}^{\text{eq}}(A) \subseteq M \models \text{VDF}_{\mathcal{EC}}$ and $p \in \mathcal{S}^{\mathcal{L}_{\partial, \text{div}}}(M)$. The following are equivalent:

- (i) p is $\mathcal{L}_{\partial, \text{div}}^{\text{eq}}(A)$ -definable;
- (ii) $\nabla_{\omega}(p)$ is $\mathcal{L}_{\text{div}}^{\text{eq}}(\mathcal{G}(A))$ -definable;
- (iii) p is $\mathcal{L}_{\partial, \text{div}}^{\text{eq}}(\mathcal{G}(A))$ -definable.

Proof. The implication (iii) \Rightarrow (i) is trivial. Let us now assume (i). An $\mathcal{L}_{\text{div}}(M)$ -formula $\varphi(\bar{x}; m)$ is in $\nabla_{\omega}(p)$ if and only if $\varphi(\partial_{\omega}(x); m) \in p$, where $\partial_{\omega}(x) = (\partial^i(x))_{i \in \omega}$. It follows that $\nabla_{\omega}(p)$ is $\mathcal{L}_{\partial, \text{div}}^{\text{eq}}(A)$ -definable. By Proposition (2.2), $\nabla_{\omega}(p)$ is in fact $\mathcal{L}_{\text{div}}^{\text{eq}}(\mathcal{G}(A))$ -definable.

Let us now assume (ii) and let $\psi(x; m)$ be any $\mathcal{L}_{\partial, \text{div}}(M)$ -formula. By quantifier elimination, $\psi(x; m)$ is equivalent to $\varphi(\partial_{\omega}(x); \partial_{\omega}(m))$ for some \mathcal{L}_{div} -formula $\varphi(\bar{x}; \bar{t})$. Therefore $\psi(x; m) \in p$ if and only if $\varphi(\bar{x}; \partial_{\omega}(m)) \in \nabla_{\omega}(p)$ and hence p is $\mathcal{L}_{\partial, \text{div}}^{\text{eq}}(\mathcal{G}(A))$ -definable. \blacksquare

In [Rid], it is shown that there are enough definable types to use this partial elimination of imaginaries result to obtain elimination of imaginaries to the geometric sorts for $\text{VDF}_{\mathcal{EC}}$. Thanks to the result in Section 1 and results from [Rid], we can also characterise invariant types in $\text{VDF}_{\mathcal{EC}}$. Note that, although the main results in [Rid] depend on the results proved in the present paper, the result from [Rid] that we will be using in what follows does not.

Proposition 2.4:

Let $M \models \text{VDF}_{\mathcal{EC}}$ and $A = \text{acl}_{\mathcal{L}_{\partial, \text{div}}}^{\text{eq}}(A) \subseteq M^{\text{eq}}$. A type $p \in \mathcal{S}^{\mathcal{L}_{\text{div}}}(M)$ is $\mathcal{L}_{\partial, \text{div}}^{\text{eq}}(A)$ -invariant if and only if it is $\mathcal{L}_{\text{div}}^{\text{eq}}(\mathcal{G}(A))$ -invariant.

2 Valued differential fields

Proof. To prove the non obvious implication, by Proposition (1.9), we have to show that $\text{VDF}_{\mathcal{EC}}$ has a model whose underlying valued field is uniformly stably embedded in any elementary extension — that is tackled in Lemma (2.1) — and that any $\mathcal{L}_{\partial, \text{div}}^{\text{eq}}(A)$ -definable set (in the sort \mathbf{K}) is consistent with an $\mathcal{L}_{\text{div}}^{\text{eq}}(\mathcal{G}(A))$ -invariant \mathcal{L}_{div} -type. It follows from [Rid, Proposition 9.7] (applied to $T = \text{ACVF}$ and $\tilde{T} = \text{VDF}_{\mathcal{EC}}$) that any $\mathcal{L}_{\partial, \text{div}}^{\text{eq}}(A)$ -definable set (in the sort \mathbf{K}) is consistent with an $\mathcal{L}_{\partial, \text{div}}^{\text{eq}}(A)$ -definable \mathcal{L}_{div} -type. But, by Proposition (2.2), such a type is $\mathcal{L}_{\text{div}}^{\text{eq}}(\mathcal{G}(A))$ -definable. ■

Corollary 2.5:

Let $A = \text{acl}_{\mathcal{L}_{\partial, \text{div}}}^{\text{eq}}(A) \subseteq M \models \text{VDF}_{\mathcal{EC}}$ and $p \in \mathcal{S}^{\mathcal{L}_{\partial, \text{div}}}(M)$. The following are equivalent:

- (i) p is $\mathcal{L}_{\partial, \text{div}}^{\text{eq}}(A)$ -invariant;
- (ii) $\nabla_{\omega}(p)$ is $\mathcal{L}_{\text{div}}^{\text{eq}}(\mathcal{G}(A))$ -invariant;
- (iii) p is $\mathcal{L}_{\partial, \text{div}}^{\text{eq}}(\mathcal{G}(A))$ -invariant.

Proof. This is proved as in Corollary (2.3), except that Proposition (2.4) is used instead of Proposition (2.2). ■

We can now give a characterisation of forking in $\text{VDF}_{\mathcal{EC}}$.

Corollary 2.6:

Let $M \models \text{VDF}_{\mathcal{EC}}$ be $|A|^+$ -saturated, $A = \text{acl}_{\mathcal{L}_{\partial, \text{div}}}^{\text{eq}}(A) \subseteq M$ and $\varphi(x)$ be an $\mathcal{L}_{\partial, \text{div}}(M)$ -formula. Then $\varphi(x)$ does not fork over A if and only if for all $\mathcal{L}_{\text{div}}(M)$ -formulas such that $\varphi(x)$ is equivalent to $\psi(\partial_{\omega}(x))$, $\psi(\bar{x})$ does not fork over $\mathcal{G}(A)$ (in ACVF).

Proof. Let us first assume that $\varphi(x)$ does not fork over A and let p be a global non forking extension of $\varphi(x)$. As $\text{VDF}_{\mathcal{EC}}$ is NIP, by [HP11, Proposition 2.1], p is invariant under all automorphisms that fix Lascar strong type over A . But, because $\text{VDF}_{\mathcal{EC}}$ has the invariant extension property (cf. [Rid, Theorem 2.14]), Lascar strong type and strong type coincide in $\text{VDF}_{\mathcal{EC}}$ (see [HP11, Proposition 2.13]), hence p is $\mathcal{L}_{\partial, \text{div}}^{\text{eq}}(A)$ -invariant. It follows from Corollary (2.5) that $\nabla_{\omega}(p)$ is $\mathcal{L}_{\text{div}}^{\text{eq}}(\mathcal{G}(A))$ -invariant and hence $\psi(\bar{x})$ does not fork over $\mathcal{G}(A)$.

Let us now assume that no $\psi(\bar{x})$ such that $\varphi(x)$ is equivalent to $\psi(\partial_{\omega}(x))$ forks over $\mathcal{G}(A)$. Then there exists $q \in \mathcal{S}_{\bar{x}}^{\mathcal{L}_{\text{div}}}(M)$ which is $\mathcal{L}_{\text{div}}^{\text{eq}}(\mathcal{G}(A))$ -invariant and consistent with all such formulas $\psi(\bar{x})$. Now, the image of the continuous map $\nabla_{\omega} : \mathcal{S}_x^{\mathcal{L}_{\partial, \text{div}}}(M) \rightarrow \mathcal{S}_{\bar{x}}^{\mathcal{L}_{\text{div}}}(M)$ is closed and if $\chi(\bar{x})$ is an $\mathcal{L}_{\text{div}}(M)$ -formula containing the image of ∇_{ω} and $\psi(\bar{x})$ is as above, $\chi(\partial_{\omega}(x)) \wedge \psi(\partial_{\omega}(x))$ is also equivalent to $\varphi(x)$. Therefore, $q = \nabla_{\omega}(p)$ for some $\mathcal{L}_{\partial, \text{div}}^{\text{eq}}(A)$ -invariant $p \in \mathcal{S}_x^{\mathcal{L}_{\partial, \text{div}}}(M)$. This type p implies $\varphi(x)$ and hence $\varphi(x)$ does not fork over A . ■

Remark 2.7:

The previous corollary is somewhat unsatisfying as one needs to consider all possible ways of describing $\varphi(x)$ as the prolongation points of an \mathcal{L}_{div} -formula ψ (with parameters in a saturated model) to conclude whether φ forks or not.

References

Considering only one such ψ cannot be enough. For example, consider any definable set $\varphi(x)$ forking (in $\text{VDF}_{\mathcal{EC}}$) over A and let $\psi(x_0, x_1) = (\text{val}(x_0) \geq 0 \wedge \text{val}(x_1) < 0) \vee \varphi(x_0)$. Then the set $\{x \in M : M \models \psi(x, \partial(x))\} = \varphi(M)$ but ψ does not fork over A (in ACVF). The obstruction here might seem frivolous, but it is the core of the problem. Indeed, it is not clear if there is a way, given φ to find a formula ψ as above that does not contain “large” subsets with no prolongation points.

References

- [HHM06] Deirdre Haskell, Ehud Hrushovski, and Dugald Macpherson. Definable sets in algebraically closed valued fields: elimination of imaginaries. *J. Reine Angew. Math.*, 597:175–236, 2006.
- [HP11] Ehud Hrushovski and Anand Pillay. On NIP and invariant measures. *J. Eur. Math. Soc. (JEMS)*, 13(4):1005–1061, 2011.
- [Mat04] Jiří Matoušek. Bounded VC-dimension implies a fractional Helly theorem. *Discrete Comput. Geom.*, 31(2):251–255, 2004.
- [Rid] Silvain Rideau. Imaginaries in valued differential fields. arXiv:1508.07935.
- [Sca00] Thomas Scanlon. A model complete theory of valued D -fields. *J. Symb. Log.*, 65(4):1758–1784, 2000.
- [Sim15a] Pierre Simon. *A Guide to NIP Theories*. Number 44 in Lect. Notes Log. Cambridge Univ. Press, Cambridge, Assoc. Symbol. Logic, Chicago, IL edition, 2015.
- [Sim15b] Pierre Simon. Invariant types in NIP theories. *J. Math. Log.*, 15(2), 2015.